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Evolution of a quantum system with supersymmetry broken by an additional potential

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Abstract. In this paper, we discuss the time evolution of a one-dimensional supersymmetric quantum mechanical system, in which the supersymmetry is broken by an additional potential. The potentials under consideration belong to a special class and can be in general time dependent. We analyse the special case of time-independent potentials and have used the Jaynes-Cummings model in quantum optics as a simple example in our analysis. For the time dependent potentials, we use the basic evolution operator technique developed by C M Cheng and P C W Fung and present a way in which to treat such problems.

1. Introduction

The symmetric nature of a quantum system is manifested by the invariant property of the Hamiltonian under the transformation representing the symmetry group, such that

$$[\hat{H}_s, \hat{g}_i] = 0 \tag{1.1}$$

where $\hat{g}_i$ is a generator of the transformation.

However, in real situations, the symmetry of a physical system is always broken by an additional potential so that the symmetry of the system is reduced. The Hamiltonian is no longer invariant under transformation in this case:

$$[\hat{H}_s, \hat{g}_i] \neq 0. \tag{1.2}$$

Supersymmetric quantum mechanics was first proposed by Witten [1] in 1981. The supersymmetric properties [1-4], the conditions for spontaneous breaking [1, 3, 5] and the applications [6-9] have been studied for the last ten years. However, the breaking of the supersymmetry by an additional potential, which becomes the theme of our paper, was seldom discussed.

In section 2, we will review supersymmetric quantum mechanics briefly and describe the irreducible subspaces: $\mathfrak{h}_0$ (one-dimensional) and $\mathfrak{h}_i$ (two-dimensional).

In section 3, we will start our discussions on symmetry breaking. The potentials considered belong to a class in which the potentials only cause the transitions among the states confined within each irreducible subspace ($\mathfrak{h}_0$ or $\mathfrak{h}_i$). In this case the potentials are well defined in each subspaces ($\mathfrak{h}_0$ or $\mathfrak{h}_i$) and can be expressed in terms of the normalized supercharge operators constructed. Here the potentials are time independent and we will find the expectation value of any observable which has eigenstates which are also the eigenstates of the supersymmetric Hamiltonian $\hat{\mathscr{H}}$ (unbroken).

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In section 4 we will give examples to illustrate our results, derived in section 3. The system under consideration is the well known Jaynes-Cummings model [10] in quantum optics. We will discuss two different forms of potentials and then evaluate the expectation values of the spin operator $\hat{S}_3$.

Section 5 is devoted to obtaining the evolution operator according to the technique presented in [11]. The time-dependent wavefunction of the system can be derived if specifications of the system are given.

Section 6 is the conclusion, and some discussion will be given.

2. Supersymmetric quantum mechanics

We first review one-dimensional supersymmetric quantum mechanics. The superalgebra involved [1, 2] is given by

$$\{\hat{Q}_i, \hat{Q}_j\} = 2\delta_{ij}\hat{\mathscr{H}} \qquad [\hat{Q}_i, \hat{\mathscr{H}}] = 0 \tag{2.1}$$

where $i, j = 1$ or 2 and the $\hat{Q}_i$ are the supercharge operators. Define the supersymmetric ladder operators:

$$\hat{Q} \equiv \tfrac{1}{2}(\hat{Q}_1 - \mathrm{i}\hat{Q}_2) \qquad \hat{Q}^{\dagger} \equiv \tfrac{1}{2}(\hat{Q}_1 + \mathrm{i}\hat{Q}_2). \tag{2.2}$$

In fact, the supercharge operators $\hat{Q}_i$ and the supersymmetric ladder operators $\hat{Q}$ and $\hat{Q}^{\dagger}$ can be written in the form

$$\hat{Q}_1 = \begin{pmatrix} 0 & \hat{A}^+ \\ \hat{A}^- & 0 \end{pmatrix} \qquad \hat{Q}_2 = \mathrm{i}\begin{pmatrix} 0 & -\hat{A}^+ \\ \hat{A}^- & 0 \end{pmatrix} \tag{2.3a}$$

$$\hat{Q} = \begin{pmatrix} 0 & 0 \\ \hat{A}^- & 0 \end{pmatrix} \qquad \hat{Q}^+ = \begin{pmatrix} 0 & \hat{A}^+ \\ 0 & 0 \end{pmatrix} \tag{2.3b}$$

where $\hat{A}^-$ is a linear differential operator and $\hat{A}^+$ is the adjoint. The Hamiltonian $\hat{\mathscr{H}}$ can then be written as

$$\hat{\mathscr{H}} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix} = \begin{pmatrix} \hat{A}^+\hat{A}^- & 0 \\ 0 & \hat{A}^-\hat{A}^+ \end{pmatrix}. \tag{2.4}$$

Thus

$$\hat{H}_1 = \hat{A}^+\hat{A}^- \qquad \text{and} \qquad \hat{H}_2 = \hat{A}^-\hat{A}^+. \tag{2.5}$$

Now we denote the set consisting of all the normalized orthogonal eigenstates with non-zero eigenvalues of $\hat{H}_1$ by $\{|\psi_1^i\rangle\}$, so that

$$\hat{H}_1|\psi_1^i\rangle = E^i|\psi_1^i\rangle \qquad \text{with } E^i > 0 \tag{2.6}$$

$$\langle\psi_1^i|\psi_1^j\rangle = \delta_{ij} \qquad \text{where } i, j = 1, 2, \ldots . \tag{2.7}$$

It can be shown that the state $\hat{A}^-|\psi_1^i\rangle$ is an eigenstate of $\hat{H}_2$ with the same eigenvalue (E^i) as $|\psi_1^i\rangle$:

$$\begin{aligned} \hat{H}_2\hat{A}^-|\psi_1^i\rangle &= \hat{A}^-\hat{A}^+\hat{A}^-|\psi_1^i\rangle \\ &= \hat{A}^-\hat{H}_1|\psi_1^i\rangle \\ &= E^i\hat{A}^-|\psi_1^i\rangle. \end{aligned} \tag{2.8}$$

Generally, $\hat{A}^-|\psi_1^i\rangle$ is not normalized. The corresponding normalized eigenstate of $\hat{H}_2$ with eigenvalue E^i is

$$|\psi_2^i\rangle = (E^i)^{-1/2}\hat{A}^-|\psi_1^i\rangle \tag{2.9}$$

since

$$\langle\psi_1^i|\hat{A}^+\hat{A}^-|\psi_1^i\rangle = E^i. \tag{2.10}$$

Conversely, $|\psi_1^i\rangle$ can be written as

$$|\psi_1^i\rangle = (E^i)^{-1/2}\hat{A}^+|\psi_2^i\rangle. \tag{2.11}$$

Hence, for any element $(|\psi_1^i\rangle)$ in $\{|\psi_1^i\rangle\}$, we can construct a normalized eigenstate of $\hat{H}_2$ denoted by $|\psi_2^i\rangle$. Both states should have same eigenvalue: $E^i > 0$. By collecting all $|\psi_2^i\rangle$, we have a set of all normalized orthogonal eigenstates of $\hat{H}_2$ with non-zero eigenvalues: $\{|\psi_2^i\rangle\}$. Such that

$$\begin{aligned}\langle\psi_2^i|\psi_2^j\rangle &= (E^i)^{-1/2}(E^j)^{-1/2}\langle\psi_1^i|\hat{A}^+\hat{A}^-|\psi_1^j\rangle\\ &= (E^i)^{-1/2}(E^j)^{1/2}\langle\psi_1^i|\psi_1^j\rangle\\ &= \delta_{ij}.\end{aligned} \tag{2.12}$$

Furthermore, we can construct a pair of orthogonal normalized states which are eigenstates of $\hat{\mathscr{H}}$ with non-zero eigenvalue (E^i):

$$\begin{pmatrix}|\psi_1^i\rangle\\ 0\end{pmatrix}: \qquad \hat{\mathscr{H}}\begin{pmatrix}|\psi_1^i\rangle\\ 0\end{pmatrix} = E^i\begin{pmatrix}|\psi_1^i\rangle\\ 0\end{pmatrix} \tag{2.13a}$$

$$\begin{pmatrix}0\\ |\psi_2^i\rangle\end{pmatrix}: \qquad \hat{\mathscr{H}}\begin{pmatrix}0\\ |\psi_2^i\rangle\end{pmatrix} = E^i\begin{pmatrix}0\\ |\psi_2^i\rangle\end{pmatrix}. \tag{2.13b}$$

The states in (2.13*a*) and (2.13*b*) can be transformed into each other:

$$\begin{pmatrix}|\psi_1^i\rangle\\ 0\end{pmatrix} = (E^i)^{-1/2}\hat{Q}^+\begin{pmatrix}0\\ |\psi_2^i\rangle\end{pmatrix} \tag{2.14a}$$

$$\begin{pmatrix}0\\ |\psi_2^i\rangle\end{pmatrix} = (E^i)^{-1/2}\hat{Q}\begin{pmatrix}|\psi_1^i\rangle\\ 0\end{pmatrix} \tag{2.14b}$$

Hence, the two-dimensional subspace $\mathfrak{h}_i$ spanned by

$$\begin{pmatrix}|\psi_1^i\rangle\\ 0\end{pmatrix} \qquad \begin{pmatrix}0\\ |\psi_2^i\rangle\end{pmatrix}$$

carries an irreducible representation of the supersymmetry.

The remaining subject that we want to discuss is the states with zero energy. It can be shown that the Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ can never have normalizable eigenstates with zero eigenvalue simultaneously [2]. Hence, the only possible normalized zero-energy ground state of $\hat{\mathscr{H}}$ is either in the form

$$\begin{pmatrix}|\psi_1^0\rangle\\ 0\end{pmatrix} \qquad \text{where } \hat{H}_1|\psi_1^0\rangle = 0 \tag{2.15a}$$

or in the form

$$\begin{pmatrix}0\\ |\psi_2^0\rangle\end{pmatrix} \qquad \text{where } \hat{H}_2|\psi_2^0\rangle = 0 \tag{2.15b}$$

but not both.

So the zero-energy ground state of $\hat{\mathscr{H}}$ spans a one-dimensional subspace $\mathfrak{h}_0$ that carries the irreducible representation of the supersymmetry. It should be noted that in the case that both $\hat{H}_1$ and $\hat{H}_2$ can have no normalizable zero-energy ground state, the supersymmetry is said to be spontaneously broken [1, 3, 5].

3. Supersymmetry broken by a time-independent potential

In this section we discuss a one-dimensional quantum system in which the supersymmetry is broken by a time-independent potential $\hat{V}$. The Hamiltonian now becomes

$$\hat{H} = \hat{\mathscr{H}} + \hat{V}. \tag{3.1}$$

The potential we considered is restricted to a class of potentials that only cause transitions to occur among the states confined within an individual subspace ($\mathfrak{h}_i$ or $\mathfrak{h}_0$). The transitions between the states in different subspaces are forbidden. Hence the potential $\hat{V}$ is well defined in each subspace. It is elementary to show that $\hat{V}$ (when defined in a two-dimensional subspace, say $\mathfrak{h}_j$ with $j \geqslant 1$) can be expressed in the form

$$\hat{V} = A_0 \hat{I} + \sum_i A_i \hat{q}_i \qquad \text{where } i = 1, 2 \text{ or } 3. \tag{3.2}$$

In the above expression, $\hat{I}$ is the identity operator, A_0 and A_i are real numbers and may not be the same in different subspaces. Then $\hat{q}_i$ are the normalized supercharge operators defined in the subspace $\mathfrak{h}_j$ ($j \geqslant 1$) by

$$\hat{q}_1 \equiv \frac{\hat{Q}_1}{(E^i)^{1/2}} \tag{3.3a}$$

$$\hat{q}_2 \equiv \frac{\hat{Q}_2}{(E^i)^{1/2}} \tag{3.3b}$$

$$\hat{q}_3 \equiv i\hat{q}_2\hat{q}_1 \tag{3.3c}$$

and it can easily be shown that the operators $\hat{q}_i$ satisfy the Clifford algebra:

$$\{\hat{q}_i, \hat{q}_j\} = 2\delta_{ij} \tag{3.4}$$

and also the su(2) algebra:

$$[\hat{q}_i, \hat{q}_j] = 2i\varepsilon_{ijk}\hat{q}_k. \tag{3.5}$$

We first confine our attention to a particular two-dimensional subspace $\mathfrak{h}_j$. By using the Heisenberg equation of motion on the normalized supercharge operators, we get (set $\hbar = 1$)

$$\dot{\hat{q}}_1 = 2(A_3\hat{q}_2 - A_2\hat{q}_3) \tag{3.6a}$$

$$\dot{\hat{q}}_2 = 2(A_1\hat{q}_3 - A_3\hat{q}_1) \tag{3.6b}$$

$$\dot{\hat{q}}_3 = 2(A_2\hat{q}_1 - A_1\hat{q}_2). \tag{3.6c}$$

After solving the above set of differential equations, we can obtain the time evolutions of the normalized supercharges:

$$\begin{aligned}\hat{q}_1(t) = [a_1^2 + (1 - a_1^2)\cos\omega t]\hat{q}_1(0) \\ + [a_1a_2(1 - \cos\omega t) + a_3 \sin\omega t]\hat{q}_2(0) \\ + [a_1a_3(1 - \cos\omega t) - a_2 \sin\omega t]\hat{q}_3(0)\end{aligned} \tag{3.7a}$$

$$\hat{q}_2(t) = [a_1 a_2(1-\cos\omega t) - a_3 \sin\omega t)\hat{q}_1(0) + [a_2^2 + (1-a_2^2)\cos\omega t]\hat{q}_2(0) + [a_2 a_3(1-\cos\omega t) + a_1 \sin\omega t]\hat{q}_3(0) \tag{3.7b}$$

$$\hat{q}_3(t) = [a_1 a_3(1-\cos\omega t) + a_2 \sin\omega t]\hat{q}_1(0) + [a_2 a_3(1-\cos\omega t) - a_1 \sin\omega t]\hat{q}_2(0) + [a_3^2 + (1-a_3^2)\cos\omega t]\hat{q}_3(0) \tag{3.7c}$$

where $a_i = A_i/\gamma$, $\gamma = (\Sigma_i (A_i)^2)^{1/2}$ and $\omega = 2\gamma$.

Now we consider an arbitrary state $|\psi\rangle$ in the subspace $\mathfrak{h}_j$ denoted by

$$\begin{pmatrix} \alpha|\psi_1\rangle \\ \beta|\psi_2\rangle \end{pmatrix}.$$

The normalization requirement is relaxed in this stage (i.e. $|\alpha|^2 + |\beta|^2$ may not be equal to 1). The expectation values of the $\hat{q}_i$ are then found to be

$$\langle\psi|\hat{q}_i(t)|\psi\rangle = B_i + C_i \cos\omega t + D_i \sin\omega t = B_i + E_i \sin(\omega t + \phi_i) \tag{3.8}$$

where

$$B_1 = 2a_1^2 \operatorname{Re}(\alpha*\beta) + 2a_1 a_2 \operatorname{Im}(\alpha*\beta) + a_1 a_3(|\alpha|^2 - |\beta|^2)$$
$$C_1 = 2(1-a_1^2)\operatorname{Re}(\alpha*\beta) - 2a_1 a_2 \operatorname{Im}(\alpha*\beta) - a_1 a_3(|\alpha|^2 - |\beta|^2)$$
$$D_1 = 2a_3 \operatorname{Im}(\alpha*\beta) - a_2(|\alpha|^2 - |\beta|^2)$$
$$B_2 = 2a_1 a_2 \operatorname{Re}(\alpha*\beta) + 2a_1^2 \operatorname{Im}(\alpha*\beta) + a_2 a_3(|\alpha|^2 - |\beta|^2)$$
$$C_2 = -2a_1 a_2 \operatorname{Re}(\alpha*\beta) + 2(1-a_2^2)\operatorname{Im}(\alpha*\beta) - a_2 a_3(|\alpha|^2 - |\beta|^2)$$
$$D_2 = -2a_3 \operatorname{Re}(\alpha*\beta) + a_1(|\alpha|^2 - |\beta|^2)$$
$$B_3 = 2a_1 a_3 \operatorname{Re}(\alpha*\beta) + 2a_2 a_3 \operatorname{Im}(\alpha*\beta) + a_3^2(|\alpha|^2 - \beta|^2)$$
$$C_3 = -2a_1 a_3 \operatorname{Re}(\alpha*\beta) - 2a_2 a_3 \operatorname{Im}(\alpha*\beta) + (1-a_3^2)(|\alpha|^2 - |\beta|^2)$$
$$D_3 = 2a_2 \operatorname{Re}(\alpha*\beta) - 2a_1 \operatorname{Im}(\alpha*\beta)$$
$$E_i = [(C_i)^2 + (D_i)^2]^{1/2}$$
$$\phi_i = \tan^{-1}(C_i/D_i).$$

It can be shown that any observable $(\hat{O})$ which is well defined in $\mathfrak{h}_j$, such that its eigenstates are also the eigenstates of the supersymmetric Hamiltonian $\hat{\mathscr{H}}$, can be expressed in the form

$$\hat{O}(t) = b_0 \hat{I} + \sum_i b_i \hat{q}_i(t) \tag{3.9}$$

where b_0 and b_i are real numbers which may be different in different subspace $\mathfrak{h}_j$. A typical example of this kind of observable is the supersymmetric Hamiltonian $\hat{\mathscr{H}}$ itself.

So

$$\langle\psi|\hat{O}(t)|\psi\rangle = b_0 + \sum_i b_i \langle\psi|\hat{q}_i(t)|\psi\rangle = \mathscr{A} + \mathscr{B}\sin(\omega t + \vartheta). \tag{3.10}$$

The last step in (3.10) is obtained by using (3.8) and the parameters involved are

$$\mathscr{A} = b_0 + \sum_i b_i B_i \qquad \mathscr{B} = \left[\left(\sum_i b_i C_i\right)^2 + \left(\sum_i b_i D_i\right)^2\right]^{1/2}$$

$$\vartheta = \tan^{-1}\left[\left(\sum_i b_i C_i\right)\left(\sum_i b_i D_i\right)^{-1}\right].$$

If we do not confine ourselves in the subspace $\mathfrak{h}_j$, we can consider an arbitrary normalized state $|\Phi\rangle$ of the supersymmetric system:

$$|\Phi\rangle = \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \end{pmatrix} = \alpha_0 \begin{pmatrix} |\psi_1^0\rangle \\ 0 \end{pmatrix} + \sum_j \begin{pmatrix} \alpha_j |\psi_1^j\rangle \\ \beta_j |\psi_2^j\rangle \end{pmatrix}$$
$$= \alpha_0 |\psi^0\rangle + \sum_j |\psi^j\rangle. \tag{3.11}$$

Without loss of generality, we assume the zero-energy ground state of $\hat{\mathscr{H}}$ is

$$|\psi^0\rangle = \begin{pmatrix} |\psi_1^0\rangle \\ 0 \end{pmatrix}$$

and the states $|\phi_1\rangle$, $|\phi_2\rangle$ are

$$|\phi_1\rangle = \alpha_0 |\psi_1^0\rangle + \sum_j \alpha_j |\psi_1^j\rangle$$

and $|\phi_2\rangle = \sum_j \beta_j |\psi_2^j\rangle$ with $j = 1, 2, \ldots$. We also denote

$$\begin{pmatrix} \alpha_j |\psi_1^j\rangle \\ \beta_j |\psi_2^j\rangle \end{pmatrix}$$

by $|\psi^j\rangle$ to specify the fact that it belongs to the subspace $\mathfrak{h}_j$. Moreover, $|\Phi\rangle$ is normalized, so that

$$|\alpha_0|^2 + \sum_j (|\alpha_j|^2 + |\beta_j|^2) = 1. \tag{3.12}$$

The expectation value of $\hat{O}$ becomes

$$\langle\Phi|\hat{O}|\Phi\rangle = |\alpha_0|^2 \langle\psi^0|\hat{O}|\psi^0\rangle + \sum_j \langle\psi^j|\hat{O}|\psi^j\rangle$$
$$= |\alpha_0|^2 \mathscr{O} + \sum_j [\mathscr{A}_j + \mathscr{B}_j \sin(\omega_j t + \vartheta_j)] \tag{3.13}$$

where $\mathscr{O}$ is the eigenvalue of $\hat{O}$ when it operates on $|\psi^0\rangle$ and the summation is obtained by using (3.10).

Equation (3.13) gives us the expectation value of any observable which shares the eigenstates with the supersymmetric Hamiltonian $\hat{\mathscr{H}}$.

4. Examples

In this section we apply our results in the last section to the well known Jaynes-Cummings model [10] in quantum optics.

Firstly, the form of Hamiltonian we adopt here is [12]

$$\hat{H}_{JC} = \hat{a}^\dagger \hat{a} + \hat{S}_3 + \lambda[S_+ \hat{a} + \hat{S}_- \hat{a}^\dagger]. \tag{4.1}$$

The system described by (4.1) can be considered to be a two-level system (e.g. a spin-$\frac{1}{2}$ particle in a static magnetic field) which interacts with a radiation field. The operator notations appearing in (4.1) have their usual meanings and λ is the coupling constant.

The Hamiltonian $\hat{H}_{JC}$ can be decomposed into two parts: the supersymmetric Hamiltonian $\hat{\mathscr{H}}$ and the applied time-independent potential $\hat{V}$. Therefore $\hat{H}_{JC} = \hat{\mathscr{H}} + \hat{V}$ with $\hat{\mathscr{H}}$ and $\hat{V}$ given by

$$\hat{\mathscr{H}} = \hat{a}^\dagger \hat{a} + \hat{S}_3 + \tfrac{1}{2}\hat{I} \tag{4.2}$$

$$\hat{V} = -\tfrac{1}{2}\hat{I} + \lambda(\hat{S}_+\hat{a} + \hat{S}_-\hat{a}^\dagger) \tag{4.3}$$

where $\hat{I}$ is the identity operator.

We can also find out the expressions of the supercharges and supersymmetric ladder operators defined in section 2:

$$\hat{Q}_1 = \hat{S}_+\hat{a} + \hat{S}_-\hat{a}^\dagger \tag{4.4a}$$

$$\hat{Q}_2 = -i(\hat{S}_+\hat{a} - \hat{S}_-\hat{a}^\dagger) \tag{4.4b}$$

$$\hat{Q} = \hat{S}_-\hat{a}^\dagger \qquad \hat{Q}^\dagger = \hat{S}_+\hat{a}. \tag{4.4c}$$

An energy eigenstate of $\hat{\mathscr{H}}$ is represented by $|n, m\rangle$ where n is the eigenvalue of $\hat{a}^\dagger\hat{a}$ and m is the eigenvalue of $\hat{S}_3$. Hence

$$\hat{\mathscr{H}}|n, m\rangle = (n + m + \tfrac{1}{2})|n, m\rangle. \tag{4.5}$$

It should be noted that n can be any non-negative integer and m can only be either $\frac{1}{2}$ or $-\frac{1}{2}$.

It is obvious that the states $|k-1, \frac{1}{2}\rangle$ and $|k, -\frac{1}{2}\rangle$ $(k \geq 1)$ have the same energy eigenvalue (k) and hence span an irreducible subspace (say $\mathfrak{h}_k$) mentioned in the last two sections.

By restriction of our attention to $\mathfrak{h}_k$ as stated in section 3, the parameters involved in (3.7a)-(3.7c) have their special forms:

$$a_1 = 1 \qquad a_2 = a_3 = 0 \tag{4.6}$$

$$\gamma = \lambda k^{1/2} \qquad \text{so } \omega = 2\gamma = 2\lambda k^{1/2}. \tag{4.7}$$

In the following we evaluate the expectation values of the operator $\hat{S}_3$ by using different initial states.

(i) Simultaneous eigenstates of $\hat{a}^\dagger\hat{a}$ and $\hat{S}_3$. The simultaneous eigenstates of $\hat{a}^\dagger\hat{a}$ and $\hat{S}_3$ are denoted by $|k-1, \frac{1}{2}\rangle$ and $|k, -\frac{1}{2}\rangle$. By using (3.10) and the fact that $\hat{S}_3(t) = \frac{1}{2}\hat{q}_3(t)$ (by setting $\hbar = 1$), we get

$$\begin{aligned}\langle k-1, \tfrac{1}{2}|\hat{S}_3(t)|k-1, \tfrac{1}{2}\rangle &= -\langle k, -\tfrac{1}{2}|\hat{S}_3(t)|k, -\tfrac{1}{2}\rangle \\ &= \tfrac{1}{2}\cos(2\lambda k^{1/2} t).\end{aligned} \tag{4.8}$$

This result represents the well known Rabi oscillations.

(ii) Coherent state. We now consider a state $|\phi\rangle$ which is an eigenstate of $\hat{S}_3$ (with eigenvalue $-\frac{1}{2}$) but not the operator $\hat{a}^\dagger\hat{a}$. The radiation field considered after (4.1) is in coherent state, such that

$$|\phi\rangle \equiv |Z\rangle|-\tfrac{1}{2}\rangle \tag{4.9}$$

where $|Z\rangle$ is the coherent state of radiation defined as

$$|Z\rangle \equiv \exp(-|Z|^2/2) \sum_{n=0}^{\infty} \frac{Z^n}{(n!)^{1/2}} |n\rangle \tag{4.10}$$

where $|Z|^2 = \bar{n}$ (mean of the photon number n) and $|n\rangle$ is the number state of the radiation field.

By using (3.13) and (4.8), we have the expectation value of $\hat{S}_3(t)$ as

$$\langle\phi|\hat{S}_3(t)|\phi\rangle = -\tfrac{1}{2}\exp(-\bar{n}) \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \cos(2\lambda n^{1/2} t). \tag{4.11}$$

Thus, by using the supersymmetric property of the system, we can obtain a result identical to that derived using conventional quantum mechanical methods. Moreover, the infinite series in (4.11) has not been evaluated in closed form, but an approximated expression was obtained [13].

Secondly, we would like to consider the modified Hamiltonian of the Jaynes-Cummings model discussed in [12]:

$$\hat{H} = \hat{a}^\dagger\hat{a} + \hat{S}_3 + \lambda[\hat{S}_+\hat{R} + \hat{S}_-\hat{R}^\dagger] \tag{4.12}$$

where $\hat{R} = \hat{a}(\hat{a}^\dagger\hat{a})^{1/2}$ and $\hat{R}^\dagger = (\hat{a}^\dagger\hat{a})^{1/2}\hat{a}^\dagger$.

The Hamiltonian $\hat{H}$ can also be decomposed into two parts: the supersymmetric Hamiltonian $\mathscr{H}$ (same as that in (4.2)) and the applied time-independent potential $\hat{V}$ given by

$$\hat{V} = -\tfrac{1}{2}\hat{I} + \lambda[\hat{S}_+\hat{R} + \hat{S}_-\hat{R}^\dagger]. \tag{4.13}$$

The system under consideration is similar to that described by (4.1), but that the interaction between the two-level system and the radiation field is intensity dependent.

Now the parameters involved in (3.7a)-(3.7c) are then found to be

$$a_1 = 1 \qquad a_2 = a_3 = 0 \tag{4.14}$$

$$\gamma = \lambda k \qquad \text{so } \omega = 2\gamma = 2\lambda k. \tag{4.15}$$

Employing the simultaneous eigenstates of $\hat{a}^\dagger\hat{a}$ and $\hat{S}_3$ as initial states, we have ($k \geqslant 1$)

$$\begin{aligned}\langle k-1, \tfrac{1}{2}|\hat{S}_3(t)|k-1, \tfrac{1}{2}\rangle &= -\langle k, -\tfrac{1}{2}|\hat{S}_3(t)|k, -\tfrac{1}{2}\rangle \\ &= \tfrac{1}{2}\cos(2\lambda k t).\end{aligned} \tag{4.16}$$

Moreover, as we use the state $|\phi\rangle$ defined in (4.9), we have

$$\langle\phi|\hat{S}_3(t)|\phi\rangle = -\tfrac{1}{2}\exp(-\bar{n}) \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \cos(2\lambda n t). \tag{4.17}$$

It is noted that equation (4.17) is an infinite sum of cosine functions, like equation (4.11). But the oscillation frequencies involved in (4.17) (proportional to n) are different from those in (4.11) (proportional to $n^{1/2}$). The expression in (4.17) can hence be summed easily to give [12]:

$$\langle\phi|\hat{S}_3(t)|\phi\rangle = -\tfrac{1}{2}\exp[-2\bar{n}\sin^2(\lambda t)]\cos[\bar{n}\sin(2\lambda t)]. \tag{4.18}$$

Equation (4.18) gives us an explicit form for the expectation value of $\hat{S}_3$ at any time t.

5. Supersymmetry broken by a time-dependent potential

In this section we consider a one-dimensional supersymmetric quantum system in which the supersymmetry is broken by a time-dependent potential. The potential considered is in the same class as well as that considered in section 3.

Since the transitions are only possible among the states confined in individual irreducible subspaces $\mathfrak{h}_i$ ($i \geq 1$) or $\mathfrak{h}_0$, we first restrict our study to a particular two-dimensional subspace $\mathfrak{h}_j$ ($j \geq 1$).

Similar to that mentioned in the time-independent case, a time-dependent potential (defined in $\mathfrak{h}_j$) can be expressed as

$$\hat{V}(t) = c_0(t)\hat{I} + \sum_i c_i(t)\hat{q}_i \tag{5.1}$$

where $\hat{I}$ is the identity operator and $i = 1, 2$ or 3. In (5.1), $c_0(t)$ and $c_i(t)$ are real-valued functions of time rather than the real numbers taken in (3.2) and may be different in different subspaces.

The Hamiltonian $\hat{H}$ that describes this system is then given by

$$\begin{aligned}\hat{H}(t) &= \hat{\mathscr{H}} + \hat{V}(t) \\ &= \hat{\mathscr{H}} + c_0(t)\hat{I} + \hat{V}'(t)\end{aligned} \tag{5.2}$$

where $\hat{\mathscr{H}}$ is the supersymmetric Hamiltonian and

$$\hat{V}'(t) = \sum_i c_i(t)\hat{q}_i.$$

Now we consider the evolution equation:

$$\hat{H}(t)\hat{U}_j(t,0) = \mathrm{i}\frac{\partial}{\partial t}\hat{U}_j(t,0) \tag{5.3}$$

where $\hat{U}_j(t,0)$ is the evolution operator of the system. Then we assume that $\hat{U}_j(t,0)$ can be decomposed into

$$\hat{U}_j(t,0) = \hat{U}'(t,0)\hat{U}_0(t,0) \tag{5.4}$$

where $\hat{U}'(t,0)$ is the solution of

$$\hat{V}'(t)\hat{U}'(t,0) = \mathrm{i}\frac{\partial}{\partial t}\hat{U}'(t,0). \tag{5.5}$$

$\hat{U}_0(t,0)$ can be determined by substitution of (5.4) into (5.3) and found to be the solution of

$$(\hat{\mathscr{H}} + c_0(t)\hat{I})\hat{U}_0(t,0) = \mathrm{i}\frac{\partial}{\partial t}\hat{U}_0(t,0)$$

so that

$$\begin{aligned}\hat{U}_0(t,0) &= \exp\left[-\mathrm{i}\left(\hat{\mathscr{H}}t + \hat{I}\int_0^t c_0(u)\,\mathrm{d}u\right)\right] \\ &= \exp\left[-\mathrm{i}\left(E^j t + \int_0^t c_0(u)\,\mathrm{d}u\right)\right]\hat{I}.\end{aligned} \tag{5.6}$$

(Note that $\hat{\mathscr{H}}|\psi^j\rangle = E^j|\psi^j\rangle$ for any state $|\psi^j\rangle$ in the Hilbert space $\mathfrak{h}_j$.)

Hence $\hat{U}_0(t,0)$ only contributes an unimportant phase factor to the wavefunction and the important operator is $\hat{U}'(t,0)$.

We have recalled that the operators $\hat{q}_i$ satisfy the Clifford and su(2) algebra. Thus they are the generators of SU(2) and then $\hat{U}'(t,0)$ can be found by the technique presented recently [11]. We now treat the problem parallel to that in [11] with only a slight modification.

We first rewrite the potential $\hat{V}'(t)$ in the form

$$\hat{V}'(t) = d_1(t)\hat{q}_+ + d_2(t)\hat{q}_3 + d_3(t)\hat{q}_- \tag{5.7}$$

where

$$\hat{q}_+ = \tfrac{1}{2}(\hat{q}_1 + \mathrm{i}\hat{q}_2) \qquad \hat{q}_- = \tfrac{1}{2}(\hat{q}_1 - \mathrm{i}\hat{q}_2)$$

$$d_1(t) = c_1(t) - \mathrm{i}c_2(t) \qquad d_2(t) = c_3(t)$$

$$d_3(t) = c_1(t) + \mathrm{i}c_2(t).$$

As mentioned in [11], the operator $\hat{U}'(t,0)$ can be expressed in the following form:

$$\hat{U}'(t,0) = \exp(g_1(t)\hat{q}_+)\exp(g_2(t)\hat{q}_3)\exp(g_3(t)\hat{q}_-). \tag{5.8}$$

By substituting (5.8) into (5.5) and comparing the coefficients on both sides, we find that the $g_i(t)$ can be determined by solving a set of differential equations:

$$\dot{g}_1(t) = f_1(t) + 2f_2(t)g_1(t) - f_3(t)g_1(t)^2 \tag{5.9a}$$

$$\dot{g}_2(t) = f_2(t) - f_3(t)g_1(t) \tag{5.9b}$$

$$\dot{g}_3(t) = f_3(t)\exp(2g_2(t)) \tag{5.9c}$$

with the initial conditions

$$g_i(0) = 0 \qquad \text{for } i = 1, 2, \text{or } 3. \tag{5.10}$$

The $f_k(t)$ are given by

$$f_k(t) = d_k(t)/\mathrm{i} \qquad \text{for } k = 1, 2 \text{ or } 3. \tag{5.11}$$

The most important part among (5.9*a*)-(5.9*c*) is (5.9*a*), which is in the form of the Riccati equation. Once it is solved, the other two equations can be solved readily:

$$g_2(t) = \int_0^t (f_2(u) - f_3(u)g_1(u))\,\mathrm{d}u \tag{5.12}$$

$$g_3(t) = \int_0^t f_3(u)\exp(2g_2(u))\,\mathrm{d}u. \tag{5.13}$$

Let us assume that the $g_i(t)$ have been found, the operator $\hat{U}'(t,0)$ is then expressible as

$$\begin{aligned}\hat{U}'(t,0) &= \exp(g_1(t)\hat{q}_+)\exp(g_2(t)\hat{q}_3)\exp(g_3(t)\hat{q}_-)\\ &= (1+g_1(t)\hat{q}_+)\exp(g_2(t)\hat{q}_3)(1+g_3(t)\hat{q}_-)\\ &= \begin{pmatrix}\exp(g_2)+g_1g_2\exp(-g_2) & g_1\exp(-g_2)\\ g_3\exp(-g_2) & \exp(-g_2)\end{pmatrix}.\end{aligned} \tag{5.14}$$

Relation (5.14) gives us the matrix form of $\hat{U}'(t,0)$ and then the explicit form of the evolution operator $\hat{U}_j(t,0)$ is

$$\begin{aligned}\hat{U}_j(t,0) = \exp&\left[-\mathrm{i}\left(E^j t + \int_0^t c_0(u)\,\mathrm{d}u\right)\right]\\ &\times\begin{pmatrix}\exp(g_2)+g_1g_2\exp(-g_2) & g_1\exp(-g_2)\\ g_3\exp(-g_2) & \exp(-g_2)\end{pmatrix}.\end{aligned} \tag{5.15}$$

In view of expression (3.11), the time evolution of $|\Phi\rangle$ is then given by

$$\begin{aligned}|\Phi(t)\rangle &= \alpha_0|\psi^0(t)\rangle + \sum_j |\psi^j(t)\rangle \\ &= \alpha_0 \exp(-\mathrm{i}\varphi)|\psi^0(0)\rangle + \sum_j \hat{U}_j(t,0)|\psi^j(0)\rangle. \end{aligned} \tag{5.16}$$

$\hat{U}_j(t,0)$ are given by equation (5.15), and the phase factor $\exp(-\mathrm{i}\varphi)$ is due to the fact that $|\psi^0(0)\rangle$ is an energy eigenstate of $\hat{H}$ defined in (5.2).

6. Conclusions

In this paper we have carried out analysis of the time evolution of a quantum system in which the supersymmetry is broken by a potential belonging to a special class which can only cause transitions among the states confined within each irreducible subspace. In fact, if the Hamiltonian of the system has supersymmetric properties, we can only obtain information about the transitions confined within $\mathfrak{h}_0$ or $\mathfrak{h}_i$ but not the transitions among the eigenstates of $\hat{H}_1$ or $\hat{H}_2$. Hence, the existence of supersymmetry in a system cannot help us to simplify the evolution problems involving the transitions among the eigenstates of $\hat{H}_1$ or $\hat{H}_2$.

In section 3 we have derived the time evolution of the expectation value of an arbitrary observable which shares its eigenstates with the supersymmetric Hamiltonian $\hat{\mathscr{H}}$.

We have shown in section 4 that we can use the 'supersymmetric technique', developed in section 3, to arrive at the important results (4.11) and (4.17) of the Jaynes–Cummings model as that obtainable via conventional methods in quantum mechanics.

In section 5, we have obtained the evolution operator for a general time-dependent potential. We can then derive the wavefunction if the system is specified.

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References

[1] Witten E 1981 *Nucl. Phys.* B **188** 513
[2] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **81** 2917
[3] Cooper F and Freedman B 1983 *Ann. Phys., NY* **146** 262
[4] Gendenshtein L E and Krive I V 1985 *Sov. Phys. Usp.* **28** 645
[5] Witten E 1982 *Nucl. Phys.* B **202** 253
[6] Blockley C A and Stedman G E 1985 *Eur. J. Phys.* **6** 218
[7] Haymaker R W and Rau A R P 1986 *Am. J. Phys.* **54** 928
[8] Kostelecky V A and Nieto M M 1984 *Phys. Rev. Lett.* **53** 2285
[9] Kostelecky V A and Nieto M M 1985 *Phys. Rev.* A **32** 1293
[10] Jaynes E T and Cummings F W 1963 *Proc. Inst. Elect. Eng.* **51** 89
[11] Cheng C M and Fung P C W 1988 *J. Phys. A: Math. Gen.* **21** 4115–31
[12] Buck B and Sukumar C V 1981 *Phys. Lett.* **81A** 132
[13] Eberly J H, Narozhny N B and Sanchez-Mondragon J J 1980 *Phys. Rev. Lett.* **44** 1323